CONVEX-OPTIMIZATION-BASED ENFORCEMENT OF ROBUST BIBO STABILITY ON THE AIC SCHEME USING A MODIFIED RLS ALGORITHM

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ABSTRACT

This paper addresses the issues relating to the enforcement of robust BIBO (ℓ_∞) stability when implementing the adaptive inverse control (AIC) scheme for noise cancelation. In this scheme, an adaptive FIR-form filter is added to a closed-loop system in order to reduce the output error caused by external disturbances. A Small-Gain-Theorem-based sufficient stability condition, which accounts for the feedback interaction between the time-varying adaptive filter and the unmodeled dynamics existing in the closed-loop plant, is derived. This condition leads to the formulation of a constrained convex optimization problem solvable recursively using a modified RLS algorithm that preserves the converge properties of the original RLS algorithm.

1. INTRODUCTION

The AIC scheme for noise cancelation [1] has been demonstrated to be very useful in a wide range of applications. In particular, it has been seen that using AIC, it is possible to enhance the performance achieved by LTI controllers employed in laser beam jitter suppression applications [2].

In general, the stability problem arises when we have a system that consists of smaller subsystems interconnected in feedback configurations. Even though each of these subsystems is internally stable, the bigger system could be unstable. In the particular case of AIC, ensuring that the adaptive filter employed is stable does not guarantee stability of the scheme as a whole. Arguments for stability of the AIC scheme have been discussed in [1]. However, those are based on conditions that are impossible to impose on a real case experiment. Such arguments do not take into account the inherent uncertainty in any identified model, due to the inability of an LTI model to capture the real dynamics of a physical system, which in general, is time-varying and nonlinear.

The analysis of robust stability and some methods for enforcing it have been addressed by this author in [3]. Here, we develop further in order to preserve the convergence properties of the recursive algorithm employed in the implementation of the adaptive filter. In the convergence analysis we follow the approach in [4].

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2. PROBLEM FORMULATION

2.1. Notation and Mathematical Preliminaries

The space of all bounded scalar-valued sequences is denoted by ℓ_∞. Thus, if $x = \{\ldots, x(-1), x(0), x(1), \ldots\}$, with $x(k) \in \mathbb{R}$, is a sequence in ℓ_∞, then

$$\|x\|_{\ell_\infty} = \sup_k |x(k)| < \infty. \tag{1}$$

From a mathematical point of view, a system is an operator that maps sequences between two signal spaces. In this case, the operators of interest are the operators that map signals from ℓ_∞ to ℓ_∞ (operators on ℓ_∞), that are linear and causal, but not necessarily time-invariant. An operator F from ℓ_∞ to ℓ_∞ is called bounded if its induced norm defined as

$$\|F\|_{\ell_\infty \to \ell_\infty} = \sup_{x \in \ell_\infty, x \neq 0} \frac{\|Fx\|_{\ell_\infty}}{\|x\|_{\ell_\infty}} \tag{2}$$

is finite. Also, an operator $F : \ell_\infty \to \ell_\infty$ is said to be stable with respect to a signal space ℓ_∞ if it is bounded on ℓ_∞ [5]. In particular, it is possible to show that the induced norm from ℓ_∞ to ℓ_∞ of the linear time-varying (LTV) system $F(k, z) = w_0(k) + w_1(k)z^{-1} + \ldots + w_{M-1}(k)z^{-M+1}$, with $w_i(k) \in \mathbb{R}$ and $k \in \mathbb{Z}$, is given by

$$\|F\|_{\ell_\infty \to \ell_\infty} = \sup_k \|w(k)\|_{\ell_1} \text{ [3].} \tag{3}$$

Next, consider the block diagram in Fig. 1. It is clear that the relations

$$\begin{aligned} y_1 &= P_2 y_2 + u_1 \\ y_2 &= P_1 y_1 + u_2 \end{aligned} \tag{4}$$

hold. The feedback connection in Fig. 1 is called well posed if (4) gives a unique output $\{y_1, y_2\}$ in for any input $\{u_1, u_2\}$ in ℓ_∞ [5]. A special case of wellposedness is given when the operator $P_1 P_2$ is strictly causal.

Lemma 1: If the operator $P_1 P_2$ is strictly causal, then the feedback connection of Fig. 1 is well posed [3], [5]. $\diamond$

The last mathematical tool that we need is a particular version of the *Small Gain Theorem*.

Theorem 1: Let $P_1 : \ell_\infty \to \ell_\infty$ and $P_2 : \ell_\infty \to \ell_\infty$ be two stable operators and assume that the closed-loop system, in Fig. 1, is well posed. Then, the closed-loop system is ℓ_∞-stable if $\|P_1\|_{\ell_\infty \to \ell_\infty} \|P_2\|_{\ell_\infty \to \ell_\infty} < 1$ [5]. $\diamond$

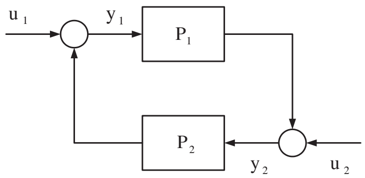

Fig. 1. Typical Feedback Connection.

2.2. The Adaptive Inverse Control Scheme

The AIC scheme is shown in Fig. 2. Here, G represents the physical system to be controlled, $\hat{G}$ is an identified LTI model of the plant G, and Q is an adaptive feedforward filter. For analysis purposes, in this subsection, we consider that $\hat{G} = G$ and $r(i) = 0$ for all i. Thus, it is clear that $\hat{n} = n$.

Now, let us define

$$n_\tau = \begin{cases} n_\tau(k) = n(k) & 0 \leq k \leq \tau \\ n_\tau(k) = 0 & otherwise \end{cases}, \tag{5}$$

let $m_\tau = Gn_\tau$, $y_\tau = v + n_\tau$, and Q_τ be an operator computed according to some law using the information at time τ. Now, recalling that SISO systems commute, it is immediate that $GQ_\tau n_\tau = Q_\tau m_\tau$. Then, the control objective is to find the operator Q_τ, which is the solution to the optimization problem

$$\min_Q \|y_\tau\|_{\ell_2} = \min_Q \|n_\tau - Qm_\tau\|_{\ell_2}. \tag{6}$$

Thus, introducing the constraint $Q(z) = \sum_{i=0}^{n-1} w_i z^{-i} z^{-1}$, the optimization problem (6) becomes

$$\min_w \|Aw - b\|_{\ell_2}^2, \tag{7}$$

where the matrices for $\tau + 1$ samples are

$$b = \begin{bmatrix} d(0) \\ d(1) \\ \vdots \\ d(\tau) \end{bmatrix} \; and \; A = \begin{bmatrix} U(0) \\ U(1) \\ \vdots \\ U(\tau) \end{bmatrix}. \tag{8}$$

Notice that $d(k) = \hat{n}(k)$ and $U(k) = [m_\tau(k-1) \; m_\tau(k-2) \; \cdots \; m_\tau(k-M)]$. It is well known that the solution to the regularized version, $\min_w \left\{\|Aw - b\|_{\ell_2}^2 + \lambda^o \|w\|_{\ell_2}^2\right\}$, of the problem (8) can be solved by the RLS recursions

$$w(k) = w(k-1) + \frac{P(k-1)U^T(k)}{1 + U(k)P(k-1)U^T(k)} e(k) \tag{9}$$

$$P(k) = P(k-1) - \frac{P(k-1)U^T(k)U(k)P(k-1)}{1 + U(k)P(k-1)U^T(k)}, \tag{10}$$

with $e(k) = [d(k) - U(k)w(k-1)]$, $w(-1) = 0$ and $P(-1) = \lambda^{o^{-1}} I$ [6]. Finally, to end this subsection, let define $F(z) = \sum_{i=0}^{M-1} w_i z^{-i}$, where M is referred as the order of the filter $F(z)$. Clearly, $Q(z) = F(z)z^{-1}$.

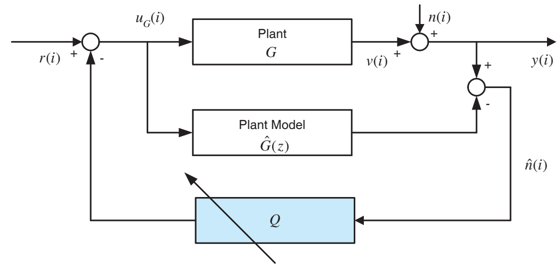

Fig. 2. Adaptive Inverse Control Scheme.

2.3. Convergence Properties of the AIC scheme using RLS

In order to analyze the convergence properties of the RLS algorithm, it has been common to define the Lyapunov-like function

$$V(k) = \tilde{w}^T(k)P^{-1}(k)\tilde{w}(k), \tag{11}$$

where, $\tilde{w}(k) = w^o - \hat{w}(k)$, being w^o the value that makes $d(k) = U(k)w^o$ [4]. Now, we review some fundamental facts about the RLS algorithm and $V(k)$. First, defining $g(k) = V(k) - V(k-1)$ and after some not-too-easy algebraical manipulations, for this case, we obtain that

$$g(k) = -\frac{\tilde{w}^T(k-1)U^T(k)U(k)\tilde{w}(k-1)}{1 + U(k)P(k-1)U^T(k)}. \tag{12}$$

Thus, from (11) and (12), we conclude that $V(k)$, is a positive, bounded, monotonically decreasing sequence and therefore $V(k)$ converges. On the other hand, using the matrix inversion lemma and the RLS recursions it is verifiable that

$$P^{-1}(k) = P^{-1}(-1) + \sum_{i=0}^{k} U^T(i)U(i). \tag{13}$$

Consequently, we can state the following.

Lemma 2: Let $\lambda_M(k)$ be the smallest eigenvalue of the positive definite matrix $P^{-1}(k)$. Thus, if $\lim_{k\to\infty} \lambda_M(k) = \infty$, then $\lim_{k\to\infty} w(k) = w^o$.

proof: Since $P^{-1}(k)$ is positive definite, it can be decomposed as $P^{-1}(k) = T(k)\Lambda(k)T^T(k)$, where $\Lambda(k)$ is a diagonal matrix with positive entries, $\lambda_1(k), \; \cdots, \; \lambda_M(k)$, in the diagonal, decreasingly ordered. Defining $\nu(k) = T^T(k)\tilde{w}(k)$, it is immediate that

$$V(k) = \sum_{i=1}^{M-1} \nu_i^2(k)\lambda_i(k) + \nu_M^2(k)\lambda_M(k).$$

$V(k)$ converges, then $[\nu_M^2(k)\lambda_M(k)]$ converges as well. Therefore, since $\lim_{k\to\infty} \lambda_M(k) = \infty$, it is immediate that $\lim_{k\to\infty} \nu_M^2(k) = 0$. Thus, $\lim_{k\to\infty} \tilde{w}(k) = 0$ and consequently $\lim_{k\to\infty} w(k) = w^o$. ⋄

By (13), it is clear that the sufficient condition of Lemma 2 is satisfied if $\lim_{k\to\infty} \lambda_{\min}[\sum_{i=0}^{k} U^T(i)U(i)] = \infty$. In the case of the AIC scheme with $F(z)$ having the FIR form, this condition is satisfied if the signal m_τ, used to form the regressor $U(i)$, is weakly persistently exciting [4] of order M.

2.4. Model Uncertainty and Robust Stability

Consider Fig. 2, and let Φ be an operator, such that, $y = \Phi r$. It is immediate that $\Phi = G$ if the condition $\hat{G} = G$ holds. Consequently, it is possible to conclude that Φ is ℓ_∞-stable for any $F(z)$ if the system plant G is stable. Unfortunately, this stability condition is not useful in a real case scenario, since, it is theoretically impossible to have a model $\hat{G}(z)$ such that $\hat{G} = G$.

Let the signals r and n in Fig. 2 be in ℓ_∞ and let $\hat{G}$, G and F be operators on ℓ_∞. In general, G is a bounded, nonlinear, time-varying and causal operator, and therefore it is natural to consider that

$$G = \hat{G} + \Delta, \tag{14}$$

where, in general Δ is a bounded, nonlinear, time-varying and causal operator, representing the differences between modeling and reality. Also, let us define $F_a(z) = -Q(z)$, and notice that F_a is strictly causal and linear. Also, notice that considering (14), the system in Fig. 2 and the system in Fig. 3 are equivalent. Hence, one can write

$$\begin{array}{l} u_G = F_a \hat{n} + r \\ \hat{n} = \Delta u_G + n \end{array}. \tag{15}$$

The development, done thus far, allows us to conclude some facts about our adaptive system. To begin, recalling Lemma 1, it is immediate that the system in Fig. 3 is well posed. Consequently, the following result also holds.

Theorem 2: Let $w(i) = [w_0(k) \; \dots \; w_{M-1}(k)]^T$ be the adaptively computed vector of gains for the prediction problem in Fig. 2 at time k. Furthermore, let $\|\Delta\|_{\ell_\infty \to \ell_\infty} < \frac{1}{\gamma}$, with $\gamma \in \mathbb{R}^{++}$. Then the system in Fig. 3 is ℓ_∞-stable if

$$\|w(k)\|_{\ell_1} \leq \gamma, \; \forall \; k = 0, 1, 2, \dots \quad [3]. \diamond \tag{16}$$

Since the systems in Fig. 2 and Fig. 3 are equivalent, Theorem 2 gives us a sufficient condition for enforcing ℓ_∞-stability on the AIC scheme in Fig. 2. Thus, (16) leads to the formulation of a new problem, namely, the constrained adaptive filtering problem, stated as

$$\min_w \|Aw - b\|_{\ell_2}^2 \quad s.t. \quad \|w\|_{\ell_1} \leq \gamma. \tag{17}$$

This optimization problem is convex, which means that an optimal solution can be found, and that this solution is unique if $rank(A) = M$. However, in this case we have the additional difficulty of finding this optimal point in a recursive manner. Thus, sometimes it could be useful to replace the constraint given by the ℓ_1-norm by the constraint $\|w\|_{\ell_2} \leq \alpha$, where, $\alpha = \frac{\gamma}{\sqrt{M}}$. This is possible, because for a vector $x \in \mathbb{R}^n$, the relationship $\|x\|_{\ell_1} \leq \sqrt{n}\|x\|_{\ell_2}$ holds.

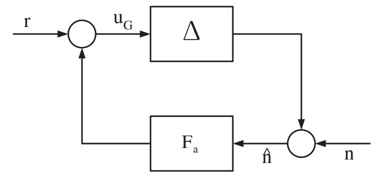

Fig. 3. Δ and F_a in Typical Feedback Connection.

3. ENFORCEMENT OF ℓ_∞-STABILITY

An exact optimal solution to (17) can be derived [3], which is recursively implementable, however, this is only applicable in a finite-time horizon and requires *a priori* knowledge about the signal m_τ that is not always available. A different approach is to find a suboptimal solution by looking for the *closest*, in some sense, feasible point to the recursively computed vector of gains, $w^{uo} = w(k)$, using the RLS algorithm. A natural *closest* point is the orthogonal projection of w^{uo} over the convex feasible region of (17) [3]. However, in order to respect the convergence properties of the RLS algorithm the following nonorthogonal projection is proposed.

$$\min_{\rho^c} \|\rho^c - \rho^{uo}\|_{\ell_2}^2 \quad s.t. \quad w^c \subset \Gamma_j, \qquad j = 1 \; or \; 2. \tag{18}$$

Where ρs and ws are related by $\rho = P^{\frac{-1}{2}} w$, with $P^{\frac{-1}{2}}$ being the Cholesky factor of $P^{-1}(k)$, i.e., $P^{-1}(k) = P^{\frac{-T}{2}} P^{\frac{-1}{2}}$. Also $\Gamma_1 = \{w^c : \|w^c\|_{\ell_1} \leq \gamma\}$ and $\Gamma_2 = \{w^c : \|w^c\|_{\ell_2} \leq \alpha\}$. When $j = 1$, we refer to this as the ℓ_1-norm constrained problem, similarly if $j = 2$, we refer to this as the ℓ_2-norm constrained problem. The problem in (18) is convex, therefore, if the true value w^o is outside the feasible region, then the projected point, $w^{co} = P^{\frac{1}{2}} \rho^{co}$ with ρ^{co} the optimal solution to (18), will stay in the border of the feasible region. On the other hand, if w^o is inside the feasible region then the following relationship immediately holds.

$$\|\rho^o - \rho^{co}\|_{\ell_2}^2 \leq \|\rho^o - \rho^{uo}\|_{\ell_2}^2. \tag{19}$$

Thus, if after computing the projection, w^{co}, we redefine $w(k)$ as $w(k) = w^{co}$, then under the assumption that w^o is feasible, the convergence properties of the original RLS algorithm remain the same, since $V(k)$ remains monotonically decreasing.

Now, we derive projections for cases the $j = 1$ and $j = 2$. If $j = 2$, the corresponding Lagrangian for (18) becomes

$$L(w^c, \mu) = (w^c - w^{uo})^T P^{-1} (w^c - w^{uo}) + \mu([w^c]^T w^c - \alpha^2)$$

and the corresponding KKT conditions are

$$\mu \geq 0 \tag{20}$$

$$\nabla_{w^c} L(w^c, \mu) = 2(P^{-1} + \mu I) w^c - 2P^{-1} w^{uo} = 0 \tag{21}$$

$$\frac{\partial L(w^c, \mu)}{\partial \mu} = [w^c]^T w^c - \alpha^2 = 0. \tag{22}$$

The KKT conditions indicate that the optimal point is given by

$$w^{co} = (P^{-1} + \mu^o I)^{-1} P^{-1} w^{uo}. \qquad (23)$$

Then replacing in (22) the following identity holds

$$[w^{uo}]^T P^{-1} (P^{-1} + \mu^o I)^{-2} P^{-1} w^{uo} = \alpha^2. \qquad (24)$$

Recalling that P^{-1} is symmetric and positive definite, then, there is a matrix Q_P, such that, $P^{-1} = Q_P \Lambda Q_P^T$, where $Q_P^T Q_P = I$, and Λ is diagonal and positive definite. Then it is immediate that

$$(P^{-1} + \mu^o I)^{-2} = Q_P \begin{bmatrix} \frac{1}{(\lambda_1+\mu^o)^2} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{(\lambda_M+\mu^o)^2} \end{bmatrix} Q_P^T.$$

Now, let $y^o = Q_P^T P^{-1} w^{uo}$. Then, μ^o solves

$$\frac{y_1^{o2}}{(\lambda_1 + \mu^o)^2} + \frac{y_2^{o2}}{(\lambda_2 + \mu^o)^2} + \ldots + \frac{{y_M^o}^2}{(\lambda_M + \mu^o)^2} = \alpha^2, \quad (25)$$

with $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_M$. Note that μ^o can be found, using a bisection-type algorithm, as the only real solution to (25).

Similarly, the case $j = 1$ can be solved, using an interior-point method, by replacing the ℓ_1-norm constraint by

$$C w^c \preceq \gamma \mathbf{1}_{2^M \times 1}, \qquad (26)$$

where C is the matrix formed by the 2^M different possible sign-vectors of w^c.

Based on the previous results, we formulate an algorithm that enforces the stability condition (16) and preserves the convergence properties of the original RLS algorithm.

Algorithm 1 (Modified RLS Algorithm):

1. Compute $w(k)$ and $P(k)$ using (9)-(10). Also, using (13), compute $P^{-1}(k)$.
2. If $\|w(k)\|_{\ell_2\ (\ell_1)} < \alpha\ (\gamma)$ set $w^{co}(k) = w(k)$, return to step 1 and compute $w(k+1)$ and $P(k+1)$. If $\|w(k)\|_{\ell_2\ (\ell_1)} \geq \alpha\ (\gamma)$ go to step 3.
3. Compute $w^{co}(k)$ according to (23)-(25) in the ℓ_2-norm case and according to a corresponding method in the ℓ_1-norm case. Set $w(k) = w^{co}(k)$ and go back to step 1.

Notice that $F(z)$ must always be implemented using $w^{co}(k)$.

4. EXPERIMENTAL IMPLEMENTATION

In order to test the ideas developed in this paper, Algorithm 1 was implemented on the MEMS/laser beam jitter experiment described in [2] using a TI TMS320C6701 Digital Signal Processor. The effectiveness of this approach is shown in Fig. 4 and Table 1. The top plot shows the time series of the measured output y for three different cases; Open-loop, the system under the action of a μ-synthesis LTI controller, and under the AIC scheme using a filter of order 15, with $\alpha = 4.0$ and $\lambda^o = 5 \times 10^{-5}$ for the case $j = 2$. Similarly, the bottom plot shows the evolution over time of the ℓ_2-norm of $w^{co}(k)$. The disturbance employed is noise with bandwidth 0-100Hz. We remark that robust stability is enforced.

Table 1. RMS values of the output error y.

Time Series	RMS values of y
Open Loop	0.3540
LTI Feedback Controller (μ-synthesis)	0.2099
Robust Adaptive Scheme	0.0477

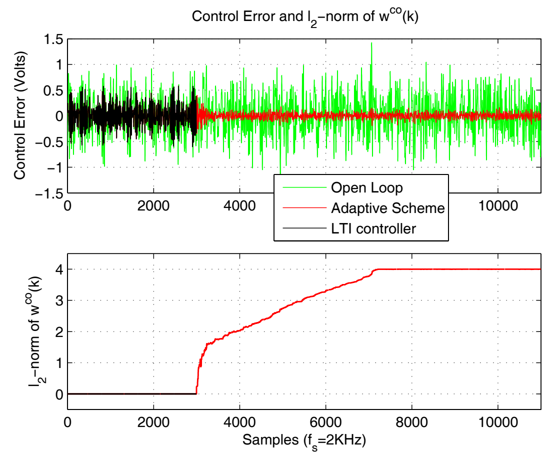

Fig. 4. Top Plot: Time Series. Bottom Plot: $\|w^{co}(k)\|_{\ell_2}$.

5. CONCLUSIONS

In this paper we derived a sufficient condition and presented a method for enforcing robust stability on the AIC scheme. This was done by formulating a convex optimization problem that can be solved recursively using a modified RLS algorithm which preserves the convergence properties of the original RLS algorithm. The effectiveness of this approach was demonstrated on a laser beam jitter control experiment.

6. REFERENCES

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